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Adjoint Monads and an Isomorphism of the Kleisli Categories

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Let $\mathfrak{A}$ be a category, $\mathbf{F}=(F, \mu, \eta)$ a monad in $\mathfrak{A}$ with multiplication μ: $F^2 \to F$ and unit η: $I \to F$, and $\mathbf{G}=(G, \delta, \varepsilon)$ a comonad in $\mathfrak{A}$ with comultiplication δ: $G \to G^2$ and counit ε: $G \to I$. The monad $\mathbf{F}$ is said to be a left adjoint of the comonad $\mathbf{G}$ [4] if there exists an adjunction β: $\mathfrak{A}(FX, Y) \simeq \mathfrak{A}(X, GY)$ such that the following two diagrams commute for all X, $Y \in \mathfrak{A}$:

$$\begin{array}{ccc} \mathfrak{A}(FX, Y) & \xrightarrow{\beta} & \mathfrak{A}(X, GY) \\ {\scriptstyle \mathfrak{A}(\eta X, Y)}\downarrow & & \downarrow{\scriptstyle \mathfrak{A}(X, \varepsilon Y)} \\ \mathfrak{A}(X, Y) & = & \mathfrak{A}(X, Y) \end{array} \tag{1}$$

$$\begin{array}{ccc} \mathfrak{A}(FX, Y) & \xrightarrow{\beta} & \mathfrak{A}(X, GY) \\ {\scriptstyle \mathfrak{A}(\mu X, Y)}\downarrow & & \downarrow{\scriptstyle \mathfrak{A}(X, \delta Y)} \\ \mathfrak{A}(F^2X, Y) & \xrightarrow{\beta^2} & \mathfrak{A}(X, G^2Y) \end{array} \tag{2}$$

Denote by $\mathfrak{A}^F$ and $\mathfrak{A}^G$ the Eilenberg–Moore categories [4, 6] of $\mathbf{F}$ and $\mathbf{G}$, respectively. Then the proof of [4, Proposition 3.3, p. 389] contains the following statement.

THEOREM 1. *If* $\mathbf{F}$ *is a left adjoint of* $\mathbf{G}$, *then the categories* $\mathfrak{A}^F$ *and* $\mathfrak{A}^G$ *are isomorphic.*

We prove a similar statement for the Kleisli categories [5, 6] $\mathfrak{A}_F$ and $\mathfrak{A}_G$ of $\mathbf{F}$ and $\mathbf{G}$, respectively.

DEFINITION 2. The monad $\mathbf{F}$ is said to be a right adjoint of the comonad $\mathbf{G}$ if there exists an adjunction α: $\mathfrak{A}(GX, Y) \simeq \mathfrak{A}(X, FY)$ such that the following two diagrams commute for all X, $Y \in \mathfrak{A}$:

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$$\begin{array}{ccc} \mathfrak{A}(GX, Y) & \xrightarrow{\alpha} & \mathfrak{A}(X, FY) \\ {\scriptstyle \mathfrak{A}(\varepsilon X, Y)} \uparrow & & \uparrow {\scriptstyle \mathfrak{A}(X, \eta Y)} \\ \mathfrak{A}(X, Y) & = & \mathfrak{A}(X, Y) \end{array} \tag{3}$$

$$\begin{array}{ccc} \mathfrak{A}(GX, Y) & \xrightarrow{\alpha} & \mathfrak{A}(X, FY) \\ {\scriptstyle \mathfrak{A}(\delta X, Y)} \uparrow & & \uparrow {\scriptstyle \mathfrak{A}(X, \mu Y)} \\ \mathfrak{A}(G^2X, Y) & \xrightarrow{\alpha^2} & \mathfrak{A}(X, F^2Y) \end{array} \tag{4}$$

THEOREM 3. *If* **F** *is a right adjoint of* **G**, *then the Kleisli categories* $\mathfrak{A}_F$ *and* $\mathfrak{A}_G$ *are isomorphic.*

Proof. The following short argument is suggested by Kleisli himself. If **T** is a monad or comonad in a category $\mathfrak{B}$, and $\mathfrak{C}$ is any category, then $\mathrm{Cat}(\mathfrak{B}_T, \mathfrak{C}) \xrightarrow{\sim} \mathrm{Cat}(\mathfrak{B}, \mathfrak{C})^{\mathrm{Cat}(T, \mathfrak{C})}$ is an obvious natural isomorphism. It is also easy to see that the functor $\mathrm{Cat}(F, \mathfrak{C})$ is a left adjoint of $\mathrm{Cat}(G, \mathfrak{C})$, and the corresponding monad and comonad in $\mathrm{Cat}(\mathfrak{A}, \mathfrak{C})$ satisfy (1) and (2). Therefore Theorem 3 follows from Theorem 1. Our original computational proof did not take notice of the latter fact. ∎

Let $\mathfrak{A}_0^F$ be the full subcategory of $\mathfrak{A}^F$ consisting of the free F-algebras, i.e., of the algebras of the form $(FX, \mu X)$ with $X \in \mathfrak{A}$. Likewise, denote by $\mathfrak{A}_0^G$ the full subcategory of $\mathfrak{A}^G$ consisting of the cofree G-coalgebras, i.e., of the coalgebras of the form $(GX, \delta X)$ with $X \in \mathfrak{A}$. It is well known [6, Exercises 1 and 2, p. 144] that $\mathfrak{A}_F$ is equivalent to $\mathfrak{A}_0^F$, and $\mathfrak{A}_G$ is equivalent to $\mathfrak{A}_0^G$.

COROLLARY 4. *If* **F** *is a right adjoint of* **G**, *then the categories* $\mathfrak{A}_0^F$ *and* $\mathfrak{A}_0^G$ *are equivalent.*

Let Γ be an associative ring with identity, K—a Γ-coring with comultiplication μ: $K \to K \otimes_\Gamma K$ and counit ε: $K \to \Gamma$. Then the endofunctor $K \otimes_\Gamma$–induces a comonad in the category Γ-Mod of left Γ-modules, and the category Induc K of cofree coalgebras is called the category of induced K-comodules. It consists of the K-comodules isomorphic to $K \otimes_\Gamma M$ for some $M \in \Gamma$-Mod. The direct summands of induced comodules are called relatively injective comodules. Likewise, for a Γ-ring S the endofunctor $S \otimes_\Gamma -$ induces a monad in Γ-Mod, and the category Induc S of free algebras, which are called induced modules, consists of all S-modules isomorphic to $S \otimes_\Gamma M$ for some $M \in \Gamma$-Mod. The direct summands of induced modules are called relatively projective modules.

COROLLARY 5. *If the Γ-coring K is finitely generated projective as a left Γ-module, then*:

(a) *The set of morphisms* $*K = \mathrm{Hom}_\Gamma(K, \Gamma)$ *in* Γ-Mod *is a* Γ-*ring finitely generated projective as a right* Γ-*module.*

(b) *The categories* Induc $*K$ *and* Induc K *are equivalent.*

Proof. (a) For $f, g \in *K$, define the product gf as the composite of the maps $K \to^{\mu} K \otimes_\Gamma K \to^{1 \otimes g} K \otimes_\Gamma \Gamma = K \to^{f} \Gamma$, and the structure map $\Gamma \to *K$—as the map sending each $\gamma \in \Gamma$ to the morphism $\gamma\varepsilon$ (remember, $*K$ is a Γ-bimodule). $*K$ is the opposite ring of the one defined in [7, 3.2, Proposition (a), (c), p. 398].

(b) For $M, N \in \Gamma$-Mod we have the natural isomorphisms of abelian groups

$$\mathrm{Hom}_\Gamma(K \otimes_\Gamma M, N) \simeq \mathrm{Hom}_\Gamma(M, \mathrm{Hom}_\Gamma(K, N)) \simeq \mathrm{Hom}_\Gamma(M, *K \otimes_\Gamma N) \quad (5)$$

because K is finitely generated projective as a left Γ-module. Hence the monad in Γ-Mod induced by the functor $*K \otimes_\Gamma -$ is a right adjoint of the comonad in Γ-Mod induced by the functor $K \otimes_\Gamma -$. Really, the commutativity of the diagrams (3) and (4) is obtained by applying the isomorphism (5), which is functorial in the Γ-bimodule K, to the morphisms ε: $K \to \Gamma$ and μ: $K \to K \otimes_\Gamma K$ of Γ-bimodules. It remains to use Corollary 4. ∎

Let Λ and A be finite-dimensional associative algebras with identity over a field R, and i: $\Lambda \to A$ an injective map of R-algebras such that Coker i, as a Λ-bimodule, is isomorphic to $\bigoplus_{s=1}^{n} I_s \otimes_R P_s$, where I_s is an injective left Λ-module, and P_s is a projective right Λ-module for all s. The isomoprhism $\mathfrak{A}_G \to \mathfrak{A}_F$ obtained in Theorem 3 and applied in the context of Corollary 5 plays an important role in the proof [2] of the existence of almost split sequences [1] in the category of finite-dimensional over R relatively projective left A-modules. Such relatively projective modules have been used [3] to establish important properties of tame finite-dimensional algebras.

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